MATH-UA 9263 - Partial Differential Equations Recitation 2: Fourier series and separation of variables

(Partial) Solutions

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January 2022

Question 1 *We consider the function shown in Fig. 1. Obtain the Fourier expansion for this periodic function.*

Solution 1 *The function is even and periodic with period $T/2$. We are thus looking for a series of the form*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right)$$

for a_0 we have

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\, dx = \frac{ah}{T}$$

and for any $n > 0$, we use

$$\begin{aligned}\int_{-T/2}^{T/2} f(x)\cos\left(\frac{2\pi n x}{T}\right)\, dx &= \int_{-T/2}^{T/2} a_n \cos^2\left(\frac{2\pi n x}{T}\right)\, dx\\ &= a_n\frac{T}{2}\end{aligned}$$

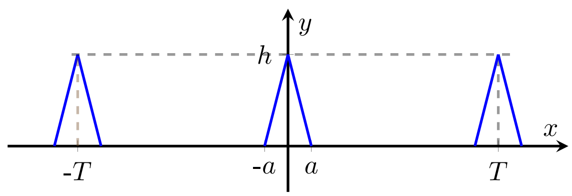

Figure 1: Question 1

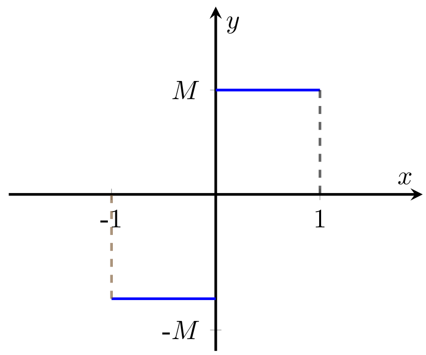

Figure 2: Question 2

Integrating by part, we get

$$
\begin{aligned}
a_n &= \frac{2}{T}\left|\frac{h}{a}(x+a)\frac{T}{2\pi n}\sin\left(\frac{2\pi nx}{T}\right)\right|_{-a}^{0} - \frac{2}{T}\int_{-a}^{0}\frac{T}{2\pi n}\sin\left(\frac{2\pi nx}{T}\right)\,dx \\
&+ \frac{2}{T}\left|\frac{h}{a}(-x+a)\frac{T}{2\pi n}\sin\left(\frac{2\pi nx}{T}\right)\right|_{0}^{a} + \frac{2}{T}\int_{0}^{a}\frac{T}{2\pi n}\sin\left(\frac{2\pi nx}{T}\right)\,dx \\
&= \frac{2}{T}\left(\frac{T}{2\pi n}\right)^2\left|\cos\frac{2\pi nx}{T}\right|_{-a}^{0} - \frac{2}{T}\left(\frac{T}{2\pi n}\right)^2\left|\cos\frac{2\pi nx}{T}\right|_{0}^{a} \\
&= \frac{2}{T}\left(\frac{T}{2\pi n}\right)^2\frac{h}{a}\left(1-\cos\frac{2\pi na}{T}\right) - \frac{2}{T}\frac{h}{a}\left(\frac{T}{2\pi n}\right)^2\left(\cos\frac{2\pi na}{T}-1\right) \\
&= \frac{2}{T}\frac{2h}{a}\left(\frac{T}{2\pi n}\right)^2\left(1-\cos\frac{2\pi an}{T}\right)
\end{aligned}
$$

Question 2 *We consider the function shown in Fig. 2. Expand this function in a complex Fourier series.*

Solution 2 *To get the expansion as a complex Fourier series, we use*

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{\frac{2\pi kx}{T}} \tag{1}$$

with

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(x) e^{-i\frac{2\pi k}{T}x}\, dx$$

Since we have

$$\int_{-T/2}^{T/2} f(x) e^{-\frac{2\pi i\ell x}{T}}\, dx = \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} c_k e^{\frac{2\pi ikx}{T}} e^{-\frac{2\pi i\ell x}{T}}\, dx$$

For $k \neq \ell$, we get

$$\begin{aligned}
\int_{-T/2}^{T/2} f(x) e^{-\frac{2\pi i\ell x}{T}}\, dx &= \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} c_k e^{\frac{2\pi ikx}{T}} e^{-\frac{2\pi i\ell x}{T}} \\
&= \sum_{k=-\infty}^{\infty} c_k \frac{T}{2\pi i(k-\ell)} \left| e^{\frac{2\pi i(k-\ell)x}{T}} \right|_{-T/2}^{T/2} \\
&= \sum_{-\infty}^{\infty} c_k \frac{T}{2\pi i(k-\ell)} \left| e^{\pi i(k-\ell)} - e^{-\pi i(k-\ell)} \right| = 0
\end{aligned}$$

if $k = \ell$, we recover

$$\int_{-T/2}^{T/2} f(x) e^{-\frac{2\pi i\ell x}{T}}\, dx = c_k T$$

hence

$$c_\ell = \frac{1}{T}\int_{-T/2}^{T/2} f(x) e^{-\frac{2\pi i\ell x}{T}}\, dx$$

From the lines above, we have

$$\begin{aligned}
c_k &= \frac{1}{T}\int_{-1}^{0} -Me^{-\frac{2\pi ikz}{T}}\, dz + \frac{1}{T}\int_{0}^{1} Me^{-\frac{2\pi ikz}{T}}\, dz \\
&= -\frac{M}{T}\frac{T}{-2\pi ik}\left(1 - e^{\frac{2\pi ik}{T}}\right) + \frac{M}{T}\frac{1}{(-2\pi ik)}\left(e^{-\frac{2\pi ik}{T}} - 1\right) \\
&= \frac{M}{\pi ik}\left(1 - \cos\left(\frac{2\pi k}{T}\right)\right)
\end{aligned}$$

Substituting into (1) and re-arranging we get

$$f(x) = \sum_{k=1}^{\infty} \frac{2M}{\pi k}\left(1 - \cos\left(\frac{2\pi k}{T}\right)\right) \sin\left(\frac{2\pi kx}{T}\right)$$

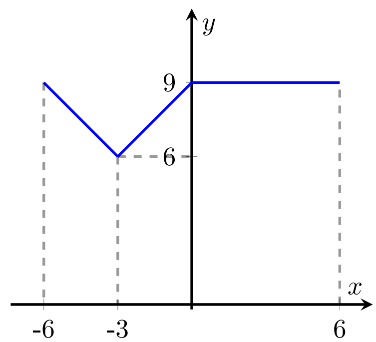

Figure 3: Question 3

Question 3 *Expand the function show in Fig. 3 into a Fourier series.*

Question 4 *A voltage $e(t) = E_0 \sin \omega t$ is passed through a half-wave rectifier which clips the negative portions of the voltage. Obtain a Fourier series expansion for the output voltage of the rectifier.*

Question 5 *Determine whether the following functions can be expanded into a Fourier series of not and give the reasons why.*

(i) $\sin \frac{1}{x}$ *defined in* $-\pi < x < \pi$

(ii) $\sin \frac{1}{x}$ *defined in* $1 < x < 2$

(iii) $\frac{1}{1+x}$ *defined in* $-2 < x < 2$

(iv) $\log x$ *defined in* $1 < x < 4$

(v) $f(x) = 1 - 2^{-(n+1)}$ *defined for* $1 - 2^{-n} < x < 1 - 2^{-(n+1)}$, $n = 0, 1, 2, \dots,$ *where* $f(x)$ *is defined in* $0 < x < 1$.

Question 6 *Establish each of the orthogonality conditions below*

$$\int_{-\pi}^{\pi} \cos mx \cos nx \; dx = \pi \delta_{m,n}$$
$$\int_{-\pi}^{\pi} \sin mx \sin nx \; dx = \pi \delta_{m,n}$$
$$\int_{-\pi}^{\pi} \sin mx \cos nx \; dx = 0$$

where δ_{mn} is the Kronecker delta defined as follows

$$\delta_{mn} = 0 \quad \text{if } m \neq n$$
$$\delta_{mn} = 1 \quad \text{if } m = n$$

Solution 6 *We use the trigonometric identity*

$$\cos mx \cos nx = \frac{\cos((m+n)x) + \cos((m-n)x)}{2}$$

From which we have

$$\int_{-\pi}^{\pi} \frac{\cos((m+n)x) + \cos((m-n)x)}{2}\, dx$$
$$= \left| \frac{\sin((m+n)x)}{2(m+n)} \right|_{-\pi}^{\pi} + \left| \frac{\sin((m-n)x)}{2(m-n)} \right|_{-\pi}^{\pi} = 0, \quad \textit{when } m \neq n \neq 0$$

$$\int_{-\pi}^{\pi} x \cos nx\, dx = 2\pi, \quad m = n = 0$$

as well as

$$\int_{-\pi}^{\pi} \cos^2 nx\, dx = \int_{-\pi}^{\pi} \frac{1 + \cos 2nx}{2}\, dx$$
$$= \pi + \int_{-\pi}^{\pi} \frac{\sin 2nx}{4n}\, dx = 0, \quad \textit{When } m = n \neq 0$$

A similar reasoning holds for

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx$$

for which using

$$\sin mx \sin nx = \frac{\cos((m-n)x) - \cos((m+n)x)}{2}$$

gives the desired result.

Finally for

$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx$$

we use

$$\sin(mx) \cos nx = \frac{\sin((m+n)x) + \sin((m-n)x)}{2}, \quad \textit{for } m \neq n \neq 0$$

From which we get

$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = \left| -\frac{\cos((m+n)x)}{2(m+n)} \right|_{-\pi}^{\pi} + \left| -\frac{\cos((m-n)x)}{2(m-n)} \right|_{-\pi}^{\pi} = 0, \quad \textit{when } m \neq n$$
$$\int_{-\pi}^{\pi} \sin mx \cos mx\, dx = 0, \quad \textit{when } m = n = 0$$
$$\int_{-\pi}^{\pi} \frac{\sin((m+n)x)}{2}\, dx = \frac{1}{2(m+n)} \left|\cos((m+n)x)\right|_{-\pi}^{\pi} = 0, \quad \textit{when } m = n \neq 0$$

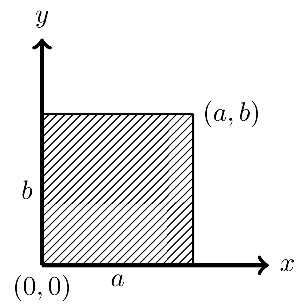

Figure 4: Question 7.

Question 7 *We consider a rectangular plate of sides a and b (see Figure 4) on three sides of which the temperature is assumed to be zero, while the temperature on the remaining side is a specified function of x, namely $f(x)$. We are thus interested in the temperature $u(x, y)$ in the plate which satisfies the (steady state) equation*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

as well as the boundary conditions

$$\begin{aligned} u(0, y) &= 0 \\ u(a, y) &= 0 \\ u(x, b) &= 0 \\ u(x, 0) &= f(x) \end{aligned}$$

Solution 7 *We start by considering solutions of the form $u(x, y) = v(x)w(y)$. Substituting this into the heat equation, we get*

$$v''(x)w(y) + w''(y)v(x) = 0$$

which we can separate into

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)}$$

As we saw during the lecture, this implies

$$\frac{v''(x)}{v(x)} = \lambda = \frac{-w''(y)}{w(y)}, \quad \textit{for some constant } \lambda \tag{2}$$

together with the boundary conditions

$$\begin{cases} v(0)w(y) = 0 \\ v(a)w(y) = 0 \\ v(x)w(b) = 0 \\ v(x)w(0) = f(x) \end{cases}$$

The relation (2) implies

- *For* $\lambda = 0$. $v(x) = Ax = b$. *Moreover, since* $v(0) = 0$ *we necessarily have* $b = 0$ *and since* $v(a) = 0$, *we have* $Aa = 0$ *which implies* $A = 0$
- *For* $\lambda = \mu^2 > 0$, *we get* $v(x) = Ae^{\mu x} + Be^{-\mu x}$. *Using* $v(0) = 0$ *gives* $A + B = 0$ *and using the condition* $v(a) = 0$ *implies* $Ae^{\mu a} - Ae^{-\mu a} = 0$ *which gives* $A = 0$
- *Finally for* $\lambda = -\mu^2 < 0$, *we get* $v(x) = Ae^{i\mu x} + Be^{-i\mu x}$. *From* $v(0) = 0$, *we get* $A + B = 0$. *From* $v(a) = 0$ *we get* $2\sin\mu a = 0$. *Together we thus have* $\mu = \frac{k\pi}{a}$, $k = 1, 2, \ldots$

Substituting the case $\lambda = -\mu_k^2$ *in the expression of* $v(x)$, *we get a collection of solutions*

$$v_k(x) = c_k \sin\frac{k\pi x}{a}, \quad k = 1, 2, \ldots \tag{3}$$

and using $\lambda = -\mu_k^2$ *in the equation for* $w(y)$, *we get*

$$\frac{w''(y)}{w(y)} = \mu_k^2 \tag{4}$$

which gives $w(y) = Ae^{\mu x} + Be^{-\mu x}$. *Using the boundary conditions, we get* $Ae^{\mu_k b} + Be^{-\mu_k b} = 0$ *so that* $B = -Ae^{2\mu_k b}$.

Combining the solutions for $v(x)$ *and* $w(y)$, *we get*

$$\begin{align} u_k(x,y) &= c_k \sin\frac{k\pi x}{a}\left(e^{\mu_k y} - e^{-\mu_k(y-2b)}\right) \tag{5} \\ &= c_k' \sin\frac{k\pi x}{a}\left(e^{\mu_k(y-b)} - e^{-\mu_k(y-b)}\right) \tag{6} \end{align}$$

Now combining the eigenfunctions, we can define our general solution as

$$u(x,y) = \sum_{k=1}^{\infty} c_k' \sin\frac{k\pi x}{a}\left(e^{\frac{-k\pi b}{a}+e^{\frac{k\pi b}{a}}}\right) = f(x), \quad \text{for } 0 < x \leq a \tag{7}$$

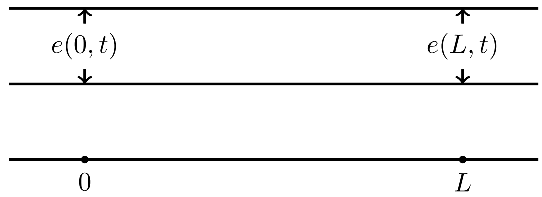

Figure 5: Question 8

So that provided that $f(x)$ can be expanded as a sum of trigonometric functions, we then have

$$c'_k = \frac{1}{\left(e^{-\frac{k\pi b}{a}} + e^{\frac{k\pi b}{a}}\right)} \int_{-a}^{a} f(x) \sin \frac{k\pi x}{a} \, dx \tag{8}$$

Since the problem is defined for $x \in (0, a)$, we can expand $f(x)$ into an odd function $\tilde{f}(x)$ and write

$$\begin{aligned} c'_k &= \frac{1}{\left(e^{-\frac{k\pi b}{a}} + e^{\frac{k\pi b}{a}}\right)} \int_{-a}^{a} \tilde{f}(x) \sin\left(\frac{k\pi x}{a}\right) \, dx \\ &= \frac{2}{\left(e^{-\frac{k\pi b}{a}} + e^{\frac{k\pi b}{a}}\right)} \int_{0}^{a} f(x) \sin\left(\frac{k\pi x}{a}\right) \, dx \end{aligned}$$

Question 8 *The voltage in a transmission line (of the submarine cable type), grounded at $x = 0$ and $x = L$, and with an initial voltage distribution $f(x)$ can be shown to satisfy the following equation*

$$\frac{\partial^2 e}{\partial x^2} = \frac{1}{K} \frac{\partial e}{\partial t}, \quad \frac{1}{K} = RC$$

with the boundary conditions

$$\begin{aligned} e(0,t) &= 0 \\ e(L,t) &= 0 \\ e(x,0) &= f(x) \end{aligned}$$

(i) *Let us assume $f(x) = E$ (constant). Find the expression of the voltage $e(x,t)$*

(ii) *We now consider a submarine cable for which the leakage conductance is very small and the frequencies are low enough to make the series inductance negligible. In this framework, the voltage $e(x,t)$ and current $i(x,t)$ in the line obey*

$$RC\frac{\partial e}{\partial t} = \frac{\partial^2 e}{\partial x^2}, \quad RC\frac{\partial i}{\partial t} = \frac{\partial^2 i}{\partial x^2}$$

where R is the series resistance in ohms per loop mile and C is the shunt capacitance in farads per mile. Solve those equations for a current and voltage in a cable of length l if at $x = 0$ and $x = l$ the cable is short-circuited (zero voltage), while the initial current distribution $i(x,0)$ is $x(l-x)$ (The relation between the current and the voltage is given by $-\frac{\partial i}{\partial x} = eG + C\frac{\partial e}{\partial t}$)

Question 9 *Another special form of the transmission line equations can be obtained by assuming that the resistance and conductance are negligible. Such an assumption which is reasonable at high frequencies gives the following system of equations for the voltage and current*

$$LC\frac{\partial^2 e}{\partial t^2} = \frac{\partial^2 e}{\partial x^2}, \quad CL\frac{\partial^2 i}{\partial t^2} = \frac{\partial^2 i}{\partial x^2}$$

which are known as the high frequency line equations. *We consider a high frequency transmission line of length ℓ which is grounded at $x = \ell$ ($e(\ell,t) = 0$) and open circuited at $x = 0$ ($i(0,t) = 0$). If the initial current and voltage distributions are $i(x,0) = I_0 \sin(7\pi/\ell)x$ and $e(x,0) = E_0 (\ell \sinh x - x \sinh \ell)$, respectively, determine the current and voltage in the line at any time t*

Question 10 *We wish to find the steady-state temperature distribution in a semicircular plate of radius a, insulated in both faces, with its curved boundary kept at a constant temperature U_0 and its bounding diameter kept at zero temperature (see Fig. 6).*

Solution 10 *Our boundary conditions in this case are given by $u(a,\theta) = U_0$ and $u(r,0) = u(r,\pi) = 0$. We use Laplace's equation in cylindrical coordinates which gives*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{9}$$

We consider solutions of the form $u(r,\theta) = v(r)w(\theta)$. Substituting this in (9), we get

$$v''(r)w(\theta) + \frac{1}{r}v'(r)w(\theta) + \frac{1}{r^2}v(r)w''(\theta) = 0 \tag{10}$$

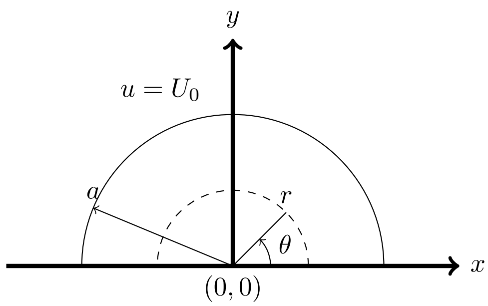

Figure 6: Question 10

Separating the variables gives

$$-\frac{w''(\theta)}{w(\theta)} = \frac{v''(r) + \frac{1}{r}v'(r)}{\frac{1}{r^2}v(r)} \tag{11}$$

We start by considering the equation in r*. For this equation, we have*

$$r^2 v''(r) + v'(r)r - \lambda v(r) = 0 \tag{12}$$

The equation

$$r^2 \frac{d^2}{dr^2 v(r)} + r\frac{dv(r)}{dr} - \lambda v(r) = 0 \tag{13}$$

is an example of a linear homogeneous differential equation with non constant coefficients. There are exceedingly few such equations that can be solved easily. Fortunately, equation (13) *is an example of such equation which is known alternatively as equidimensional, Cauchy or Euler equation. The simplest way to solve this equation is to note that for the linear differential operator*

$$(r^2\frac{d^2}{dr^2} + r\frac{d}{dr} - \lambda),$$

any power $v(r) = r^p$ *reproduces itself (as a comparison, for constant coefficients linear differential operators, it is the exponential function that reproduces itself).*

If we substitute $v(r) = r^p$ *in* (13)*, we indeed get*

$$p(p-1)r^p + pr^p - \lambda r^p = 0$$
$$(p(p-1) + p - \lambda)r^p = 0$$

We thus get two distinct solutions $p = \pm\mu$ *where* $\mu = \sqrt{\lambda}$ *except when* $\lambda = 0$ *in which case we have*

$$r^2\frac{d^2}{dr^2}v(r) + r\frac{d}{dr}v(r) = 0 = (rv(r))' \Rightarrow v'(r) = \frac{C}{r}$$

which gives $v(r) = C\log r + D$.

We can thus consider two families of solutions:

$$v(r) = c_1 r^{\mu} + c_2 r^{-\mu} \tag{14}$$
$$v(r) = \bar{c}_1 \log r + \bar{c}_2, \quad \textit{when } \mu = 0 \tag{15}$$

Also to see why when $\lambda = 0$, $v(r)$ *is the general solution, note that we can always use the change of variable* $r = e^s$ *(*r *is always non negative), from which we can view the function* $v(r)$ *as a function in* s *and derive the equation*

$$\begin{aligned}
&r^2\frac{d^2v}{dr^2} + rv'(r) - \lambda v(r)\\
&= r^2\left(\frac{dv}{ds}\left(\frac{d^2s}{dr^2}\right) + \frac{d^2v}{ds^2}\left(\frac{ds}{dr}\right)^2\right) + r\frac{dv}{ds}\frac{ds}{dr} - \lambda v\\
&= r^2\frac{dv}{ds}\left(-\frac{1}{r^2}\right) + \frac{d^2v}{ds^2} + \frac{dv}{ds} - \lambda v\\
&= \frac{d^2v}{ds^2} - \lambda v = 0
\end{aligned}$$

Hence $v(s) = Ae^s + Be^{-s}$ *provided that* $\lambda > 0$ *which we can always assume as otherwise the* θ *equation* $w''(\theta)/w(\theta) = -\lambda$ *would imply exponential solutions for* θ, $w(\theta) = Ae^{\mu\theta} + Be^{-\mu\theta}$ *which does not make sense with respect to the physics.*

We also see that our BC $u(r, 0)$, $r \in (0, a)$ *prevents the* $\log r$ *solution which would approach* ∞ *as* $r \to 0$ *so we are left with* $\mu = \lambda^2$ *and a solution in* r *of the form*

$$v(r) = c_1 r^{\mu} + c_2 r^{-\mu} \tag{16}$$

the condition $u(r, 0) = 0$ *also immediately gives* $c_2 = 0$ *as again a non zero* c_2 *would imply that* $u \to \infty$ *as* $r \to 0$.

For θ*, we get* $w''(\theta)/w(\theta) = -\lambda$ *hence* $w(\theta) = Ae^{\mu i\theta} + Be^{-i\mu\theta}$*. Again,* $u(r, 0) = 0$ *and* $u(r, \pi) = 0$ *gives* $w(0) = A + B = 0$ *and* $\sin \pi\mu = 0$ *which finally give* $\mu = k$, $k = 1, 2, \ldots$

Grouping our solutions in r *and* θ*, we get*

$$U(r, \theta) = c_1 r^k \sin k\pi\theta, \quad k = 1, 2, \ldots \tag{17}$$

to enforce our last boundary condition, we combine the eigenfunctions as

$$U(r, \theta) = \sum_{k=1}^{\infty} c_{1,k} r^k \sin k\theta, \quad \textit{on } 0, \pi \tag{18}$$

Applying the last condition, we thus get

$$U(a,\theta) = \sum_{k=1}^{\infty} c_{a,k} a^k \sin k\theta = U_0 \tag{19}$$

the difficulty here stems from the fact that we required $U(r,0) = 0$ so we can't just add a constant to the Fourier series. we must express the constant with the Fourier series. We now use $\tilde{U}_0$ to denote the (odd) extension $\tilde{U}_0$ of U_0 (i.e. $\tilde{U}_0 = U_0$ on $[0,\pi]$ and $-U_0$ on $[-\pi,0]$).

Multiplying both sides of (19) by $\sin k\pi\theta$ and integrating, we get

$$\begin{aligned}
\int_{-\pi}^{\pi} \tilde{U}_0 \sin k\theta &= c_{1,k} a^k \int_{-\pi}^{\pi} \sin^2 k\theta \, d\theta \\
&= c_{1,k} a^k \int_{-\pi}^{\pi} \frac{1-\cos 2k\theta}{2} \\
&= c_{1,k} a^k \pi
\end{aligned}$$

Finally

$$c_{1,k} = \frac{2}{\pi a^k} \int_0^{\pi} U_0 \sin k\theta \, d\theta = \begin{cases} \dfrac{2U_0}{\pi a^k} \left| \dfrac{\cos k\theta}{k} \right|_0^{\pi} = \dfrac{4U_0}{k\pi a^k} & \text{When } k \text{ is odd} \\ 0 & \text{When } k \text{ is even} \end{cases}$$

Question 11 *Solve the one dimensional heat equation*

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

with the adiabatic boundary conditions

$$\begin{aligned}
&\frac{\partial}{\partial x} u(0,t) = 0 \\
&\frac{\partial}{\partial x} u(L,t) = 0 \\
&u(x,0) = x
\end{aligned}$$

Question 12 *Solve the heat equation for one-dimensional transient flow with the boundary conditions*

$$\begin{aligned}
u(0,t) &= 0 \\
u(L,t) &= 0 \\
u(x,0) &= x(L-x)
\end{aligned}$$

Question 13 *A ring-shaped plate (see Fig. 7) of inner radius a and outer radius b is insulated on its lateral surfaces.*

1. *Find the steady state temperature $u(r, \theta)$ if the initial temperature on the inner circle is $A\theta(2\pi - \theta)$ and the initial distribution on the outer circle is $B\theta^2(2\pi - \theta)$. $A, B < \infty$ are both constants*
2. *Investigate the solution as the inner radius approaches 0.*

Solution 13 *Using the Laplacian in cylindrical coordinates, we get*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

we let $u(r, \theta) = v(r)w(\theta)$ and substitute this expression in the steady state heat equation which gives

$$v''(r)w(\theta) + \frac{1}{r}v'(r)w(\theta) + \frac{1}{r^2}v(r)w''(\theta) = 0$$

Separating the variables, we recover the Euler/Cauchy equation in r and the usual eigenvalue problem in θ,

$$\left(v''(r) + \frac{1}{r}v'(r)\right)w(\theta) = -\frac{1}{r^2}v(r)w''(\theta)$$

Hence

$$\frac{r^2 v''(r) + rv'(r)}{v(r)} = -\frac{w''(\theta)}{w(\theta)} = \lambda$$

introducing the parameter $\lambda = \mu^2$, and letting $r = e^s$ the solution of the Euler/Cauchy equation are given by

$$\begin{cases} v(r) = C_1 r^{\mu} + C_2 r^{-\mu} & \text{When } \lambda > 0 \\ v(r) = A_1 \log(r) + A_2 & \text{When } \lambda = 0 \end{cases}$$

Note that a solution in $\lambda < 0$ would give solutions that would be oscillating in r and exponential in θ which would not make sense in terms of the physics.

The case $\lambda = 0$ leads to solution in θ of the form $w(\theta) = A\theta + B$ which reduce to a multiplicative constant if we use $u(0) = u(2\pi)$ and $\frac{\partial u}{\partial \theta}(0) = \frac{\partial u}{\partial \theta}(2\pi)$.

The case $\lambda > 0$ gives the usual complex exponentials (or sine and cosine functions if we focus on real solutions),

$$w(\theta) = A\cos\mu\theta + B\sin\mu\theta \tag{20}$$

Using the boundary conditions at 0 and 2π we can immediately reduce those solutions to

$$w(\theta) = A\sin(k\theta)$$

for $k \in \mathbb{Z}$ since we want our solutions to be 2π-periodic.

All in all, we thus get the general solution

$$u(r,\theta) = A_1 \log(r) + A_2 + \sum_{k=1}^{\infty} \left(c_{1,k} r^k + c_{2,k} r^{-k}\right) \sin k\theta$$

We fix the remaining constants by expanding our boundary conditions into their respective Fourier series,

$$u(a,\theta) = A\theta(2\pi - \theta) = A_1 \log(a) + A_2 + \sum_{k=1}^{\infty} \left(c_{1,k} a^k + c_{2,k} a^{-k}\right) \sin k\theta$$
$$u(b,\theta) = B\theta^2(2\pi - \theta) = A_1 \log(b) + A_2 + \sum_{k=1}^{\infty} \left(c_{1,k} b^k + c_{2,k} b^{-k}\right) \sin k\theta$$

Using the orthogonality of the trigonometric functions, we get

$$A_1 \log(a) + A_2 = \frac{1}{2\pi} \int_0^{2\pi} A\theta(2\pi - \theta)\, d\theta$$
$$A_1 \log(b) + A_2 = \frac{1}{2\pi} \int_0^{2\pi} B\theta^2(2\pi - \theta)\, d\theta$$

as well as

$$c_{1,k} a^k + c_{2,k} a^{-k} = \frac{1}{\pi} \int_0^{2\pi} A\theta(2\pi - \theta) \sin k\theta\, d\theta$$
$$c_{1,k} b^k + c_{2,k} b^{-k} = \frac{1}{\pi} \int_0^{2\pi} B\theta^2(2\pi - \theta) \sin k\theta\, d\theta$$

Question 14 *Solve the heat equation*

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

subject to the boundary conditions

(i) $u(0,t) = A$ *(a constant)*

(ii) $u(L,t) = B$ *(a constant)*

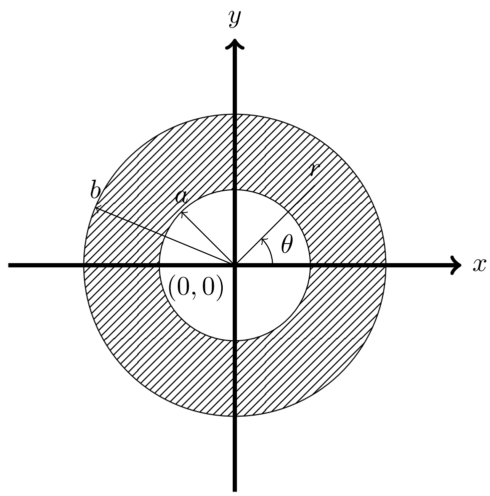

Figure 7: Question 13.

(iii) $u(x,0) = x^2(L-x)$

Question 15 *Solve the heat equation* $\partial u/\partial t = k\partial^2 u/\partial x^2$, $0 < x < L$, $t > 0$, *subject to*

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0$$
$$\frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0$$

(a) $u(x,0) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$

(b) $u(x,0) = 6 + 4\cos\dfrac{3\pi x}{L}$

(c) $u(x,0) = -2\sin\dfrac{\pi x}{L}$

(d) $u(x,0) = -3\cos\dfrac{8\pi x}{L}$

Question 16 *Consider the heat equation with a known source* $q(x,t)$*:*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t) \quad \textit{with} \quad u(0,t) = 0 \quad \textit{and} \quad u(L,t) = 0$$

Assume that $q(x,t)$ *(for each* $t > 0$*) is a picewise smooth function of* x*. Also assume that* u *and* $\partial u/\partial x$ *are continuous functions of* x *(for* $t > 0$*) and* $\partial^2 u/\partial x^2$ *and* $\partial u/partialt$ *are piecewise smooth. Thus,*

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L}$$

What ordinary differential equation does b_n satisfy? Do not solve this differential equation.

Question 17 *Consider the non homegeneous heat equation (with a steady heat source):*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + g(x)$$

Solve this equation with the initial condition

$$u(x,0) = f(x)$$

and the boundary conditions

$$u(0,t) = 0 \quad \textit{and} \quad u(L,t) = 0$$

Assume that a continuous solution exists (with continuous derivatives). [Hints: Expand the solution as a Fourier sine series (i.e. use the eigenfunction expansion). Expand $g(x)$ as a Fourier sine series. Solve for the Fourier sine series of the solution. Justify all differentiation with respect to x.]

References

[1] Richard Haberman, *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, Fourth Edition, Pearson 2004.

[2] Kenneth S. Miller, *Partial Differential Equations in Engineering Problems*, Dover Publications inc. 2020.