MATH-UA 9263 - Partial Differential Equations
PSet 2: Laplace equation, Harmonic functions

Augustin Cosse

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Given date: March 2
Due date: March 20
Total: 25pts

Question 1 (5pts) *Solve Laplace's equation inside the rectangle* $0 \leq x \leq L$, $0 \leq y \leq H$ *with the boundary conditions*

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, H) = f(x).$$

Question 2 (5pts, Schwartz reflection principle) *Let*

$$B_1^+ = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1, \quad y > 0\right\}$$

and $u \in C^2(B_1^+) \cap C(\overline{B_1^+})$ *harmonic in* B_1^+, $u(x, 0) = 0$. *Show that the function*

$$U(x, y) = \begin{cases} u(x, y) & y \geq 0 \\ -u(x, -y) & y < 0 \end{cases}$$

obtained by odd reflection with respect to y *is harmonic in all of* B_1. *[Hint: You can assume and use the uniqueness of the solution to the Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

for any bounded domain Ω *and function* $u \in C^2(\Omega) \cap C(\overline{\Omega})$*]*

Question 3 (5pts) *Let* $f \in C^2(\mathbb{R}^2)$ *with compact support* K *and*

$$u(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |\boldsymbol{x} - \boldsymbol{y}| f(\boldsymbol{y})\, d\boldsymbol{y}$$

Show that

$$u(\boldsymbol{x}) = -\frac{M}{2\pi} \log |\boldsymbol{x}| + O(|\boldsymbol{x}|^{-1}), \quad \text{as } |\boldsymbol{x}| \to +\infty$$

where $M = \int_{\mathbb{R}^2} f(\boldsymbol{y})\, d\boldsymbol{y}$ *[Hint: write* $\log |\boldsymbol{x} - \boldsymbol{y}| = \log(|\boldsymbol{x} - \boldsymbol{y}|/|\boldsymbol{x}|) + \log |\boldsymbol{x}|$ *and show that, if* $\boldsymbol{y} \in K$ *then* $|\log(|\boldsymbol{x} - \boldsymbol{y}|/|\boldsymbol{x}|)| \leq C/|\boldsymbol{x}|$*]*

Question 4 (5pts) *Find the Green functions for the following domains:*

1. *The half plane* $\{(x, y) \mid x > c\}$
2. *The disk* $\{(x, y) \mid \|(x, y) - (c_1, c_2)\| < R\}$

Question 5 (5pts) *Let* $U \in \mathbb{R}^N$ *be a bounded open set. We say that a function* u *is harmonic on* U *if* $u \in C^2(U)$ *and* $\Delta u = 0$ *on* U*. We say that* $v \in C^2(\overline{U})$ *is* subharmonic *on* U *iff* $-\Delta v \leq 0$ *in* U*.*

(a) Prove that if u *is real harmonic, the zeros of* u *are never isolated.*

(b) Let $\phi \in C^\infty(\mathbb{R})$ *be convex (*$\phi''(x) \geq 0$*) and let* u *be harmonic in* U*. Prove that the composition* $\phi(u)$ *is subharmonic*

(c) Let u *be harmonic in* U*. Prove that* $|\nabla u|^2$ *is subharmonic.*

References

[1] Richard Haberman, *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, Fourth Edition, Pearson 2004.

[2] Kenneth S. Miller, *Partial Differential Equations in Engineering Problems*, Dover Publications inc. 2020.

[3] Walter A. Strauss, *Partial Differential Equations An Introduction*, John Wiley and Sons Ltd, 2008

[4] Sandro Salsa, *Partial Differential Equations in Action, From Modelling to Theory*, Springer, 2016.