MATH-UA 9263 - Partial Differential Equations Recitation 2: Fourier series and separation of variables

Augustin Cosse

January 2022

Question 1 *We consider the function shown in Fig. 1. Obtain the Fourier expansion for this periodic function.*

Question 2 *We consider the function shown in Fig. 2. Expand this function in a complex Fourier series.*

Question 3 *Expand the function show in Fig. 3 into a Fourier series.*

Question 4 *A voltage* $e(t) = E_0 \sin \omega t$ *is passed through a half-wave rectifier which clips the negative portions of the voltage. Obtain a Fourier series expansion for the output voltage of the rectifier.*

Question 5 *Determine whether the following functions can be expanded into a Fourier series of not and give the reasons why.*

(i) $\sin \frac{1}{x}$ *defined in* $-\pi < x < \pi$

(ii) $\sin \frac{1}{x}$ *defined in* $1 < x < 2$

(iii) $\frac{1}{1+x}$ *defined in* $-2 < x < 2$

(iv) $\log x$ *defined in* $1 < x < 4$

(v) $f(x) = 1 - 2^{-(n+1)}$ *defined for* $1 - 2^{-n} < x < 1 - 2^{-(n+1)}$, $n = 0, 1, 2, \ldots,$ *where* $f(x)$ *is defined in* $0 < x < 1$.

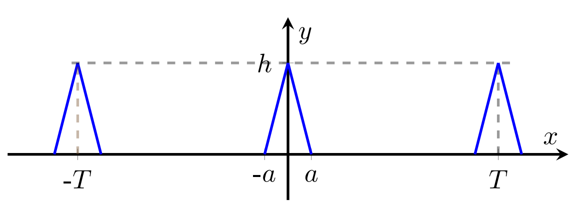

Figure 1: Question 1

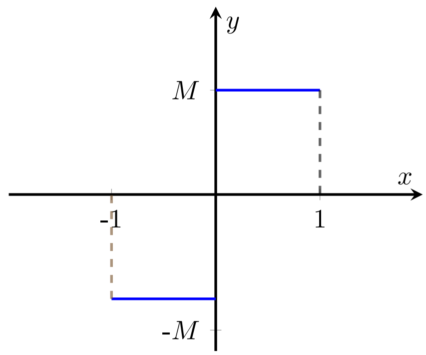

Figure 2: Question 2

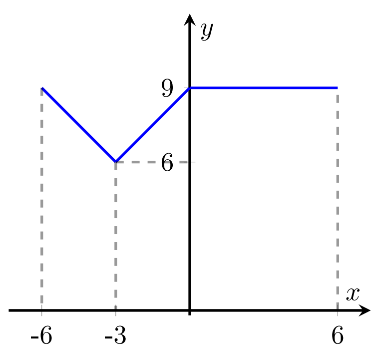

Figure 3: Question 3

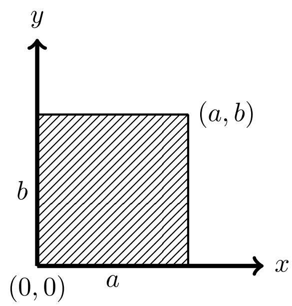

Figure 4: Question 7.

Question 6 *Establish each of the orthogonality conditions below*

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \pi \delta_{m,n}$$
$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \pi \delta_{m,n}$$
$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

where δ_{mn} is the Kronecker delta defined as follows

$$\delta_{mn} = 0 \quad \text{if } m \neq n$$
$$\delta_{mn} = 1 \quad \text{if } m = n$$

Question 7 *we consider a rectangular plate of sides a and b (see Figure 4) on three sides of which the temperature is assumed to be zero, while the temperature on the remaining side is a specified function of x, namely $f(x)$. We are thus interested in the temperature $u(x, y)$ in the plate which satisfies the (steady state) equation*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

as well as the boundary conditions

$$u(0, y) = 0$$
$$u(a, y) = 0$$
$$u(x, b) = 0$$
$$u(x, 0) = f(x)$$

Question 8 *The voltage in a transmission line (of the submarine cable type), grounded at $x = 0$ and $x = L$, and with an initial voltage distribution $f(x)$ can*

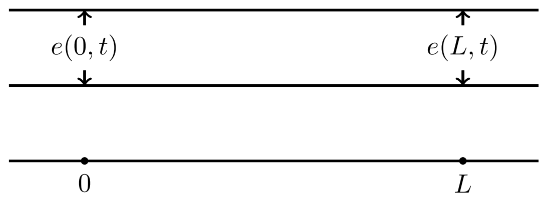

Figure 5: Question 8

be shown to satisfy the following equation

$$\frac{\partial^2 e}{\partial x^2} = \frac{1}{K}\frac{\partial e}{\partial t}, \quad \frac{1}{K} = RC$$

with the boundary conditions

$$\begin{aligned} e(0,t) &= 0 \\ e(L,t) &= 0 \\ e(x,0) &= f(x) \end{aligned}$$

(i) *Let us assume $f(x) = E$ (constant). Find the expression of the voltage $e(x,t)$*

(ii) *We now consider a submarine cable for which the leakage conductance is very small and the frequencies are low enough to make the series inductance negligible. In this framework, the voltage $e(x,t)$ and current $i(x,t)$ in the line obey*

$$RC\frac{\partial e}{\partial t} = \frac{\partial^2 e}{\partial x^2}, \quad RC\frac{\partial i}{\partial t} = \frac{\partial^2 i}{\partial x^2}$$

where R is the series resistance in ohms per loop mile and C is the shunt capacitance in farads per mile. Solve those equations for a current and voltage in a cable of length l if at $x = 0$ and $x = l$ the cable is short-circuited (zero voltage), while the initial current distribution $i(x,0)$ is $x(l-x)$ (The relation between the current and the voltage is given by $-\frac{\partial i}{\partial x} = eG + C\frac{\partial e}{\partial t}$)

Question 9 *Another special form of the transmission line equations can be obtained by assuming that the resistance and conductance are negligible. Such an assumption which is reasonable at high frequencies gives the following system of equations for the voltage and current*

$$LC\frac{\partial^2 e}{\partial t^2} = \frac{\partial^2 e}{\partial x^2}, \quad CL\frac{\partial^2 i}{\partial t^2} = \frac{\partial^2 i}{\partial x^2}$$

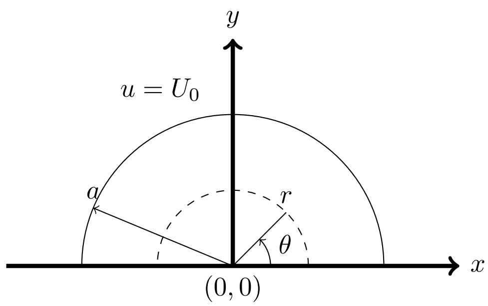

Figure 6: Question 10

which are known as the high frequency line equations*. We consider a high frequency transmission line of length ℓ which is grounded at $x = \ell$ ($e(\ell, t) = 0$) and open circuited at $x = 0$ ($i(0, t) = 0$). If the initial current and voltage distributions are $i(x, 0) = I_0 \sin(7\pi/\ell)x$ and $e(x, 0) = E_0 (\ell \sinh x - x \sinh \ell)$, respectively, determine the current and voltage in the line at any time t*

Question 10 *We wish to find the steady-state temperature distribution in a semicircular plate of radius a, insulated in both faces, with its curved boundary kept at a constant temperature U_0 and its bounding diameter kept at zero temperature (see Fig. 6).*

Question 11 *Solve the one dimensional heat equation*

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

with the adiabatic boundary conditions

$$\begin{aligned} &\frac{\partial}{\partial x} u(0, t) = 0 \\ &\frac{\partial}{\partial x} u(L, t) = 0 \\ &u(x, 0) = x \end{aligned}$$

Question 12 *Solve the heat equation for one-dimensional transient flow with the boundary conditions*

$$\begin{aligned} &u(0, t) = 0 \\ &u(L, t) = 0 \\ &u(x, 0) = x(L - x) \end{aligned}$$

Question 13 *A ring-shaped plate (see Fig. 7 below) of inner radius a and outer radius b is insulated on its lateral surfaces.*

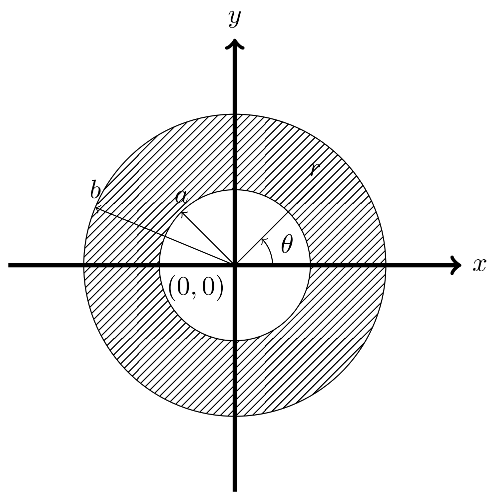

Figure 7: Question 13.

1. *Find the steady state temperature* $u(r, \theta)$ *if the initial temperature on the inner circle is* $A\theta(2\pi - \theta)$ *and the initial distribution on the outer circle is* $B\theta^2(2\pi - \theta)$. $A, B < \infty$ *are both constants*

2. *Investigate the solution as the inner radius approaches* 0.

Question 14 *Solve the heat equation*

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

subject to the boundary conditions

(i) $u(0, t) = A$ *(a constant)*

(ii) $u(L, t) = B$ *(a constant)*

(iii) $u(x, 0) = x^2(L - x)$

Question 15 *Solve the heat equation* $\partial u/\partial t = k\partial^2 u/\partial x^2$, $0 < x < L$, $t > 0$, *subject to*

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad t > 0$$
$$\frac{\partial u}{\partial x}(L, t) = 0, \quad t > 0$$

(a) $u(x,0) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$

(b) $u(x,0) = 6 + 4\cos\frac{3\pi x}{L}$

(c) $u(x,0) = -2\sin\frac{\pi x}{L}$

(d) $u(x,0) = -3\cos\frac{8\pi x}{L}$

Question 16 *Consider the heat equation with a known source* $q(x,t)$*:*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t) \quad \textit{with} \quad u(0,t) = 0 \quad \textit{and} \quad u(L,t) = 0$$

Assume that $q(x,t)$ *(for each* $t > 0$*) is a picewise smooth function of* x*. Also assume that* u *and* $\partial u/\partial x$ *are continuous functions of* x *(for* $t > 0$*) and* $\partial^2 u/\partial x^2$ *and* $\partial u/partialt$ *are piecewise smooth. Thus,*

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L}$$

What ordinary differential equation does b_n *satisfy? Do not solve this differential equation.*

Question 17 *Consider the non homegeneous heat equation (with a steady heat source):*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + g(x)$$

Solve this equation with the initial condition

$$u(x,0) = f(x)$$

and the boundary conditions

$$u(0,t) = 0 \quad \textit{and} \quad u(L,t) = 0$$

Assume that a continuous solution exists (with continuous derivatives). [Hints: Expand the solution as a Fourier sine series (i.e. use the eigenfunction expansion). Expand $g(x)$ as a Fourier sine series. Solve for the Fourier sine series of the solution. Justify all differentiation with respect to x.]

References

[1] Richard Haberman, *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, Fourth Edition, Pearson 2004.

[2] Kenneth S. Miller, *Partial Differential Equations in Engineering Problems*, Dover Publications inc. 2020.