MATH-UA 9263 - Partial Differential Equations
PSet 1: Fourier and separation of variables

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Given date: February 9
Due date: February 21
Total: 20pts

Question 1 (3pts) *According to the Dirichlet's test, if a function $f(x)$ of period 2π is piecewise monotonic on a segment $[-\pi, \pi]$ and has a finite number of discontinuities there, then its Fourier series is convergent to $f(x_0)$ at every point of continuity and to the sum $S_0 = \frac{1}{2}[f(x_{0,+} + f(x_{0,-})]$ at every point of discontinuity. Provide evidence for the Dirichlet's test by computing the coefficients of the Fourier series (is it a sine or a cosine series?) of the function shown in Fig. 1 below. Plot the Fourier series for the first few terms (plot the series consisting of the first term, first two, three and first 10 terms) on top of the function on $[-\pi, \pi]$ using your favorite language (matlab, python, julia,..)*

Question 2 (4pts) *Solve the following initial-boundary value problem*

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1, t > 0 \\ u(0,t) = 1, & t \geq 0 \\ u(1,t) = 2, & t \geq 0 \\ u(x,0) = 1 + x + 2\sin(\pi x) & \end{cases}$$

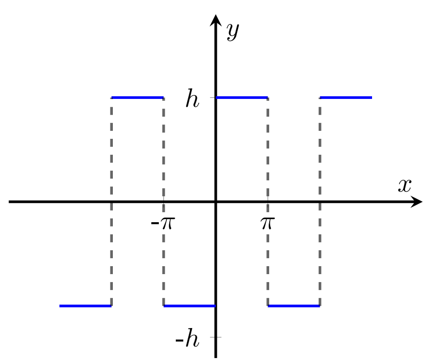

Figure 1: The square wave considered in question 1.

Question 3 (4pts, Salsa) *Use the method of separation of variables to solve the following nonhomegeneous initial-Neumann problem*

$$\begin{cases} u_t - u_{xx} = tx & 0 < x < L, t > 0 \\ u(x,0) = 1 & 0 \le x \le L \\ u_x(0,t) = u_x(L,t) = 0 & t > 0 \end{cases}$$

[Hint: write the candidate solution as $u(x,t) = \sum_{k \ge 0} c_k(t) v_k(x)$ *where* v_k *are the eigenfunctions of the eigenvalue problem associated with the homogeneous equation.]*

Question 4 (4pts, Salsa) *Use the method of separation of variables to solve (at least formally) the following mixed problem*

$$\begin{cases} u_t - Du_{xx} = 0 & 0 < x < L, t > 0 \\ u(x,0) = g(x) & 0 \le x \le L \\ u_x(0,t) = 0 & t > 0 \\ u_x(L,t) + u(L,t) = U & t > 0 \end{cases}$$

Question 5 (5pts, Strauss) *Consider the diffusion equation* $u_t = u_{xx}$ *in* $\{0 < x < 1, 0 < t < \infty\}$ *with* $u(0,t) = u(1,t) = 0$ *and* $u(x,0) = 4x(1-x)$.

(a) Show that $0 < u(x,t) < 1$ *for all* $t > 0$ *and* $0 < x < 1$.

(b) Show that $u(x,t) = u(1-x,t)$ *for all* $t \ge 0$ *and* $0 \le x \le 1$

(c) Use the energy method to show that $\int_0^1 u^2 \, dx$ *is a strictly decreasing function of* t.

References

[1] Richard Haberman, *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, Fourth Edition, Pearson 2004.

[2] Kenneth S. Miller, *Partial Differential Equations in Engineering Problems*, Dover Publications inc. 2020.

[3] Walter A. Strauss, *Partial Differential Equations An Introduction*, John Wiley and Sons Ltd, 2008

[4] Sandro Salsa, *Partial Differential Equations in Action, From Modelling to Theory*, Springer, 2016.